

Module 3 : Analysis of Strain

3.1.1 INTRODUCTION

To define normal strain, refer to the following Figure 3.1 where line AB of an axially loaded member has suffered deformation to become $A'B'$.

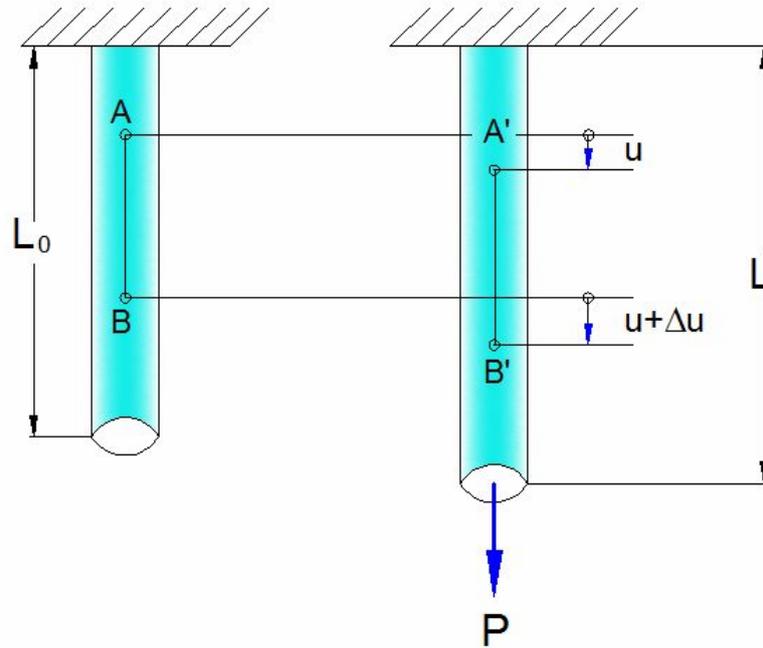


Figure 3.1 Axially loaded bar

The length of AB is Δx . As shown in Figure 3.1(b), points A and B have each been displaced, i.e., at point A an amount u , and at point B an amount $u + \Delta u$. Point B has been displaced by an amount Δu in addition to displacement of point A , and the length Δx has been increased by Δu . Now, normal strain may be defined as

$$\epsilon_x = \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} = \frac{du}{dx} \quad (3.0)$$

In view of the limiting process, the above represents the strain at a point. Therefore "Strain is a measure of relative change in length, or change in shape".

3.1.2 TYPES OF STRAIN

Strain may be classified into direct and shear strain.

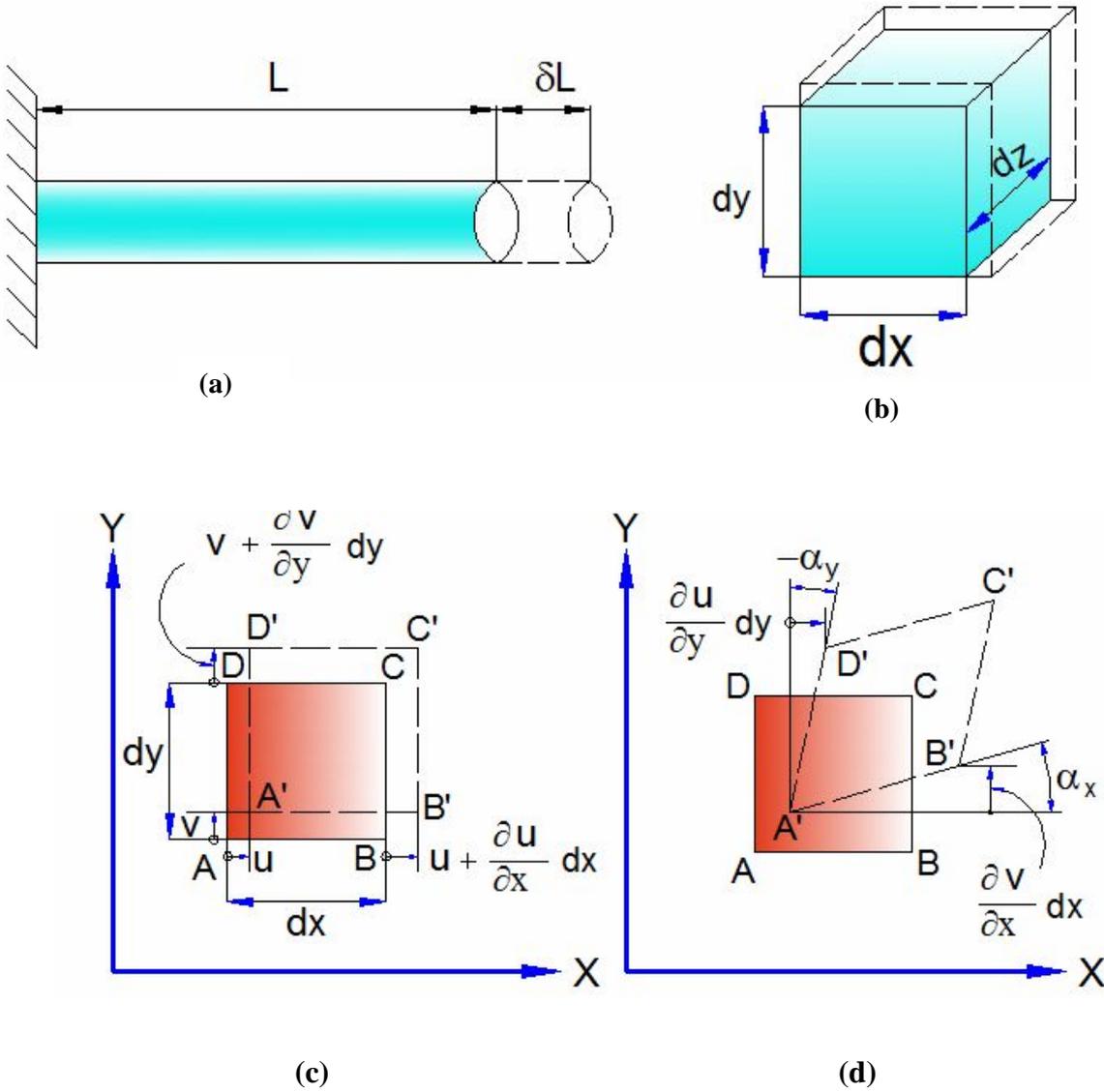


Figure 3.2 Types of strains

Figure 3.2(a), 3.2(b), 3.2(c), 3.2(d) represent one-dimensional, two-dimensional, three-dimensional and shear strains respectively.

In case of two-dimensional strain, two normal or longitudinal strains are given by

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y} \quad (3.1)$$

+ve sign applies to elongation; -ve sign, to contraction.

Now, consider the change experienced by right angle DAB in the Figure 3.2 (d). The total angular change of angle DAB between lines in the x and y directions, is defined as the shearing strain and denoted by γ_{xy} .

$$\therefore \gamma_{xy} = \alpha_x + \alpha_y = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad (3.2)$$

The shear strain is positive when the right angle between two positive axes decreases otherwise the shear strain is negative.

In case of a three-dimensional element, a prism with sides dx , dy , dz as shown in Figure 3.2(c) the following are the normal and shearing strains:

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z} \quad (3.3)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

The remaining components of shearing strain are similarly related:

$$\gamma_{xy} = \gamma_{yx}, \quad \gamma_{yz} = \gamma_{zy}, \quad \gamma_{zx} = \gamma_{xz} \quad (3.4)$$

3.1.3 DEFORMATION OF AN INFINITESIMAL LINE ELEMENT

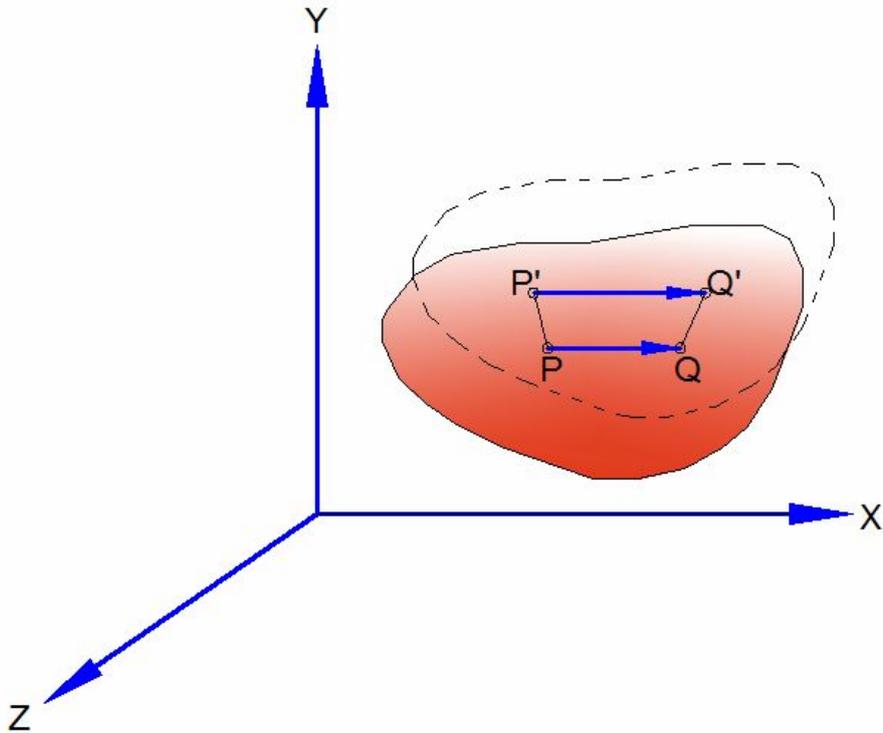


Figure 3.3 Line element in undeformed and deformed body

Consider an infinitesimal line element PQ in the undeformed geometry of a medium as shown in the Figure 3.3. When the body undergoes deformation, the line element PQ passes into the line element $P'Q'$. In general, both the length and the direction of PQ are changed.

Let the co-ordinates of P and Q before deformation be $(x, y, z), (x + \Delta x, y + \Delta y, z + \Delta z)$ respectively and the displacement vector at point P have components (u, v, w) .

The co-ordinates of P, P' and Q are

$$P : (x, y, z)$$

$$P' : (x + u, y + v, z + w)$$

$$Q : (x + \Delta x, y + \Delta y, z + \Delta z)$$

The displacement components at Q differ slightly from those at point P since Q is away from P by $\Delta x, \Delta y$ and Δz .

\therefore The displacements at Q are

$$u + \Delta u, v + \Delta v \text{ and } w + \Delta w$$

Now, if Q is very close to P , then to the first order approximation

$$\Delta u = \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y + \frac{\partial u}{\partial z} \Delta z \quad (\text{a})$$

$$\text{Similarly, } \Delta v = \frac{\partial v}{\partial x} \Delta x + \frac{\partial v}{\partial y} \Delta y + \frac{\partial v}{\partial z} \Delta z \quad (\text{b})$$

$$\text{And } \Delta w = \frac{\partial w}{\partial x} \Delta x + \frac{\partial w}{\partial y} \Delta y + \frac{\partial w}{\partial z} \Delta z \quad (\text{c})$$

The co-ordinates of Q' are, therefore,

$$Q'(x + \Delta x + u + \Delta u, y + \Delta y + v + \Delta v, z + \Delta z + w + \Delta w)$$

Before deformation, the segment PQ had components $\Delta x, \Delta y$ and Δz along the three axes.

After deformation, the segment $P'Q'$ has components $\Delta x + u, \Delta y + v$ and $\Delta z + w$ along the three axes.

Here the terms like $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$ etc. are important in the analysis of strain. These are the

gradients of the displacement components in x, y and z directions. These can be represented in the form of a matrix called the displacement-gradient matrix such as

$$\begin{bmatrix} \frac{\partial u_i}{\partial x_j} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}$$

3.1.4 CHANGE IN LENGTH OF A LINEAR ELEMENT

When the body undergoes deformation, it causes a point $P(x, y, z)$ in the body under consideration to be displaced to a new position P' with co-ordinates $(x + u, y + v, z + w)$ where u, v and w are the displacement components. Also, a neighbouring point Q with co-ordinates $(x + \Delta x, y + \Delta y, z + \Delta z)$ gets displaced to Q' with new co-ordinates $(x + \Delta x + u + \Delta u, y + \Delta y + v + \Delta v, z + \Delta z + w + \Delta w)$.

Now, let ΔS be the length of the line element PQ with its components $(\Delta x, \Delta y, \Delta z)$.

$$\therefore (\Delta S)^2 = (PQ)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

Similarly, $\Delta S'$ be the length $P'Q'$ with its components

$$(\Delta x' = \Delta x + \Delta u, \Delta y' = \Delta y + \Delta v, \Delta z' = \Delta z + \Delta w)$$

$$\therefore (\Delta S')^2 = (P'Q')^2 = (\Delta x + \Delta u)^2 + (\Delta y + \Delta v)^2 + (\Delta z + \Delta w)^2$$

From equations (a), (b) and (c),

$$\Delta x' = \left(1 + \frac{\partial u}{\partial x}\right) \Delta x + \frac{\partial u}{\partial y} \Delta y + \frac{\partial u}{\partial z} \Delta z$$

$$\Delta y' = \frac{\partial v}{\partial x} \Delta x + \left(1 + \frac{\partial v}{\partial y}\right) \Delta y + \frac{\partial v}{\partial z} \Delta z$$

$$\Delta z' = \frac{\partial w}{\partial x} \Delta x + \frac{\partial w}{\partial y} \Delta y + \left(1 + \frac{\partial w}{\partial z}\right) \Delta z$$

Taking the difference between $(\Delta S')^2$ and $(\Delta S)^2$, we get

$$\begin{aligned} (P'Q')^2 - (PQ)^2 &= (\Delta S')^2 - (\Delta S)^2 \\ &= \left\{ \left((\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2 \right) - \left((\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \right) \right\} \\ &= 2 \left(\varepsilon_x \Delta x^2 + \varepsilon_y \Delta y^2 + \varepsilon_z \Delta z^2 + \varepsilon_{xy} \Delta x \Delta y + \varepsilon_{yz} \Delta y \Delta z + \varepsilon_{zx} \Delta x \Delta z \right) \end{aligned} \quad (3.5)$$

where

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial x} \right)^2 \right] \quad (3.5a)$$

$$\varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2} \left[\left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right] \quad (3.5b)$$

$$\varepsilon_z = \frac{\partial w}{\partial z} + \frac{1}{2} \left[\left(\frac{\partial u}{\partial z} \right)^2 + \left(\frac{\partial v}{\partial z} \right)^2 + \left(\frac{\partial w}{\partial z} \right)^2 \right] \quad (3.5c)$$

$$\varepsilon_{xy} = \varepsilon_{yx} = \left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right] \quad (3.5d)$$

$$\varepsilon_{yz} = \varepsilon_{zy} = \left[\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} + \frac{\partial u}{\partial y} \frac{\partial u}{\partial z} + \frac{\partial v}{\partial y} \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \frac{\partial w}{\partial z} \right] \quad (3.5e)$$

$$\varepsilon_{zx} = \varepsilon_{xz} = \left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} \frac{\partial v}{\partial x} + \frac{\partial w}{\partial z} \frac{\partial w}{\partial x} \right] \quad (3.5f)$$

Now, introducing the notation

$$\varepsilon_{PQ} = \frac{\Delta S' - \Delta S}{\Delta S}$$

which is called the relative extension of point P in the direction of point Q , now,

$$\begin{aligned} \frac{(\Delta S')^2 - (\Delta S)^2}{2} &= \left(\frac{\Delta S' - \Delta S}{\Delta S} + \frac{(\Delta S' - \Delta S)^2}{2(\Delta S)^2} \right) (\Delta S)^2 \\ &= \left[\varepsilon_{PQ} + \frac{1}{2} (\varepsilon_{PQ})^2 \right] (\Delta S)^2 \\ &= \varepsilon_{PQ} \left[1 + \frac{1}{2} \varepsilon_{PQ} \right] (\Delta S)^2 \end{aligned}$$

From Equation (3.5), substituting for $(\Delta S')^2 - (\Delta S)^2$, we get

$$\varepsilon_{PQ} \left(1 + \frac{1}{2} \varepsilon_{PQ} \right) (\Delta S)^2 = \varepsilon_x (\Delta x)^2 + \varepsilon_y (\Delta y)^2 + \varepsilon_z (\Delta z)^2 + \varepsilon_{xy} \Delta x \Delta y + \varepsilon_{yz} \Delta y \Delta z + \varepsilon_{zx} \Delta x \Delta z$$

If l , m , and n are the direction cosines of PQ , then

$$l = \frac{\Delta x}{\Delta S}, \quad m = \frac{\Delta y}{\Delta S}, \quad n = \frac{\Delta z}{\Delta S}$$

Substituting these quantities in the above expression,

$$\varepsilon_{PQ} \left(1 + \frac{1}{2} \varepsilon_{PQ} \right) = \varepsilon_x l^2 + \varepsilon_y m^2 + \varepsilon_z n^2 + \varepsilon_{xy} lm + \varepsilon_{yz} mn + \varepsilon_{zx} nl$$

The above equation gives the value of the relative displacement at point P in the direction PQ with direction cosines l , m and n .

3.1.5 CHANGE IN LENGTH OF A LINEAR ELEMENT-LINEAR COMPONENTS

It can be observed from the Equation (3.5a), (3.5b) and (3.5c) that they contain linear terms like $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}, \dots$ etc., as well as non-linear terms like

$\left(\frac{\partial u}{\partial x}\right)^2, \left(\frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y}\right), \dots$ etc. If the deformation imposed on the body is small, the terms like $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \dots$ etc are extremely small so that their squares and products can be neglected.

Hence retaining only linear terms, the linear strain at point P in the direction PQ can be obtained as below.

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z} \quad (3.6)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \quad (3.6a)$$

$$\text{and } \varepsilon_{PQ} \cong \varepsilon_{PQ} = \varepsilon_x l^2 + \varepsilon_y m^2 + \varepsilon_z n^2 + \gamma_{xy} lm + \gamma_{yz} mn + \gamma_{zx} nl \quad (3.6b)$$

If however, the line element is parallel to x axis, then $l = 1, m = 0, n = 0$ and the linear strain is

$$\varepsilon_{PQ} = \varepsilon_x = \frac{\partial u}{\partial x}$$

Similarly, for element parallel to y axis, then $l = 0, m = 1, n = 0$ and the linear strain is

$$\varepsilon_{PQ} = \varepsilon_y = \frac{\partial v}{\partial y}$$

and for element parallel to z axis, then $l = 0, m = 0, n = 1$ and the linear strain is

$$\varepsilon_{PQ} = \varepsilon_z = \frac{\partial w}{\partial z}$$

The relations expressed by equations (3.6) and (3.6a) are known as the strain displacement relations of Cauchy.

3.1.6 STRAIN TENSOR

Just as the state of stress at a point is described by a nine-term array, the strain can be represented tensorially as below:

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (i, j = x, y, z) \quad (3.7)$$

The factor 1/2 in the above Equation (3.7) facilitates the representation of the strain transformation equations in indicial notation. The longitudinal strains are obtained when $i = j$; the shearing strains are obtained when $i \neq j$ and $\varepsilon_{ij} = \varepsilon_{ji}$.

It is clear from the Equations (3.2) and (3.3) that

$$\varepsilon_{xy} = \frac{1}{2} \gamma_{xy}, \quad \varepsilon_{yz} = \frac{1}{2} \gamma_{yz}, \quad \varepsilon_{xz} = \frac{1}{2} \gamma_{xz} \quad (3.8)$$

Therefore the strain tensor ($\varepsilon_{ij} = \varepsilon_{ji}$) is given by

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_x & \frac{1}{2} \gamma_{xy} & \frac{1}{2} \gamma_{xz} \\ \frac{1}{2} \gamma_{yx} & \varepsilon_y & \frac{1}{2} \gamma_{yz} \\ \frac{1}{2} \gamma_{zx} & \frac{1}{2} \gamma_{zy} & \varepsilon_z \end{bmatrix} \quad (3.9)$$

3.1.7 STRAIN TRANSFORMATION

If the displacement components u , v and w at a point are represented in terms of known functions of x , y and z respectively in cartesian co-ordinates, then the six strain components can be determined by using the strain-displacement relations given below.

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \quad \text{and} \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

If at the same point, the strain components with reference to another set of co-ordinates axes x' , y' and z' are desired, then they can be calculated using the concepts of axis transformation and the corresponding direction cosines. It is to be noted that the above equations are valid for any system of orthogonal co-ordinate axes irrespective of their orientations.

Hence

$$\varepsilon_{x'} = \frac{\partial u}{\partial x'}, \quad \varepsilon_{y'} = \frac{\partial v}{\partial y'}, \quad \varepsilon_{z'} = \frac{\partial w}{\partial z'}$$

$$\gamma_{x'y'} = \frac{\partial u}{\partial y'} + \frac{\partial v}{\partial x'}, \quad \gamma_{y'z'} = \frac{\partial v}{\partial z'} + \frac{\partial w}{\partial y'}, \quad \gamma_{z'x'} = \frac{\partial w}{\partial x'} + \frac{\partial u}{\partial z'}$$

Thus, the transformation of strains from one co-ordinate system to another can be written in matrix form as below:

$$\begin{bmatrix} \varepsilon_{x'} & \frac{1}{2}\gamma_{x'y'} & \frac{1}{2}\gamma_{x'z'} \\ \frac{1}{2}\gamma_{y'x'} & \varepsilon_{y'} & \frac{1}{2}\gamma_{y'z'} \\ \frac{1}{2}\gamma_{z'x'} & \frac{1}{2}\gamma_{z'y'} & \varepsilon_{z'} \end{bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \times \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_z \end{bmatrix} \times \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}$$

In general, $[\varepsilon'] = [a][\varepsilon][a]^T$

3.1.8 SPHERICAL AND DEVIATORIAL STRAIN TENSORS

Like the stress tensor, the strain tensor is also divided into two parts, the spherical and the deviatorial as,

$$E = E'' + E'$$

$$\text{where } E'' = \begin{bmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{bmatrix} = \text{spherical strain} \quad (3.10)$$

$$E' = \begin{bmatrix} (\varepsilon_x - e) & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & (\varepsilon_y - e) & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{xy} & (\varepsilon_z - e) \end{bmatrix} = \text{deviatorial strain} \quad (3.11)$$

$$\text{and } e = \frac{\varepsilon_x + \varepsilon_y + \varepsilon_z}{3}$$

It is noted that the spherical component E'' produces only volume changes without any change of shape while the deviatorial component E' produces distortion or change of shape. These components are extensively used in theories of failure and are sometimes known as "dilatation" and "distortion" components.

3.1.9 PRINCIPAL STRAINS - STRAIN INVARIANTS

During the discussion of the state of stress at a point, it was stated that at any point in a continuum there exists three mutually orthogonal planes, known as Principal planes, on which there are no shear stresses.

Similar to that, planes exist on which there are no shear strains and only normal strains occur. These planes are termed as principal planes and the corresponding strains are known as Principal strains. The Principal strains can be obtained by first determining the three mutually perpendicular directions along which the normal strains have stationary values. Hence, for this purpose, the normal strains given by Equation (3.6b) can be used.

$$\text{i.e., } \varepsilon_{PQ} = \varepsilon_x l^2 + \varepsilon_y m^2 + \varepsilon_z n^2 + \gamma_{xy} lm + \gamma_{yz} mn + \gamma_{zx} nl$$

As the values of l , m and n change, one can get different values for the strain ε_{PQ} . Therefore, to find the maximum or minimum values of strain, we are required to equate $\frac{\partial \varepsilon_{PQ}}{\partial l}$, $\frac{\partial \varepsilon_{PQ}}{\partial m}$, $\frac{\partial \varepsilon_{PQ}}{\partial n}$ to zero, if l , m and n were all independent. But, one of the direction cosines is not independent, since they are related by the relation.

$$l^2 + m^2 + n^2 = 1$$

Now, taking l and m as independent and differentiating with respect to l and m , we get

$$\begin{aligned} 2l + 2n \frac{\partial n}{\partial l} &= 0 \\ 2m + 2n \frac{\partial n}{\partial m} &= 0 \end{aligned} \tag{3.12}$$

Now differentiating ε_{PQ} with respect to l and m for an extremum, we get

$$0 = 2l\varepsilon_x + m\gamma_{xy} + n\gamma_{zx} + \frac{\partial n}{\partial l} (l\gamma_{zx} + m\gamma_{zy} + 2n\varepsilon_z)$$

$$0 = 2m\varepsilon_y + l\gamma_{xy} + n\gamma_{yz} + \frac{\partial n}{\partial m} (l\gamma_{zx} + m\gamma_{zy} + 2n\varepsilon_z)$$

Substituting for $\frac{\partial n}{\partial l}$ and $\frac{\partial n}{\partial m}$ from Equation 3.12, we get

$$\frac{2l\varepsilon_x + m\gamma_{xy} + n\gamma_{zx}}{l} = \frac{l\gamma_{zx} + m\gamma_{zy} + 2n\varepsilon_z}{n}$$

$$\frac{2m\varepsilon_y + l\gamma_{xy} + n\gamma_{yz}}{m} = \frac{l\gamma_{zx} + m\gamma_{zy} + 2n\varepsilon_z}{n}$$

Denoting the right hand expression in the above two equations by 2ε ,

$$\begin{aligned} 2\varepsilon_x l + \gamma_{xy} m + \gamma_{xz} n - 2\varepsilon l &= 0 \\ \gamma_{xy} l + 2\varepsilon_y m + \gamma_{yz} n - 2\varepsilon m &= 0 \end{aligned} \quad (3.12a)$$

$$\text{and } \gamma_{zx} l + \gamma_{zy} m + 2\varepsilon_z n - 2\varepsilon n = 0$$

Using equation (3.12a), we can obtain the values of l , m and n which determine the direction along which the relative extension is an extremum. Now, multiplying the first Equation by l , the second by m and the third by n , and adding them,

We get

$$2(\varepsilon_x l^2 + \varepsilon_y m^2 + \varepsilon_z n^2 + \gamma_{xy} lm + \gamma_{yz} mn + \gamma_{zx} nl) = 2\varepsilon(l^2 + m^2 + n^2) \quad (3.12b)$$

$$\text{Here } \varepsilon_{PQ} = \varepsilon_x l^2 + \varepsilon_y m^2 + \varepsilon_z n^2 + \gamma_{xy} lm + \gamma_{yz} mn + \gamma_{zx} nl$$

$$l^2 + m^2 + n^2 = 1$$

Hence Equation (3.12b) can be written as

$$\varepsilon_{PQ} = \varepsilon$$

which means that in Equation (3.12a), the values of l , m and n determine the direction along which the relative extension is an extremum and also, the value of ε is equal to this extremum. Hence Equation (3.12a) can be written as

$$\begin{aligned} (\varepsilon_x - \varepsilon)l + \frac{1}{2}\gamma_{xy}m + \frac{1}{2}\gamma_{xz}n &= 0 \\ \frac{1}{2}\gamma_{yx}l + (\varepsilon_y - \varepsilon)m + \frac{1}{2}\gamma_{yz}n &= 0 \\ \frac{1}{2}\gamma_{zx}l + \frac{1}{2}\gamma_{zy}m + (\varepsilon_z - \varepsilon)n &= 0 \end{aligned} \quad (3.12c)$$

Denoting,

$$\frac{1}{2}\gamma_{xy} = \varepsilon_{xy}, \quad \frac{1}{2}\gamma_{yz} = \varepsilon_{yz}, \quad \frac{1}{2}\gamma_{zx} = \varepsilon_{zx} \quad \text{then}$$

Equation (3.12c) can be written as

$$\begin{aligned}
(\varepsilon_x - \varepsilon) + \varepsilon_{xy}m + \varepsilon_{xz}n &= 0 \\
\varepsilon_{yx}l + (\varepsilon_y - \varepsilon)m + \varepsilon_{yz}n &= 0 \\
\varepsilon_{zx}l + \varepsilon_{zy}m + (\varepsilon_z - \varepsilon)n &= 0
\end{aligned} \tag{3.12d}$$

The above set of equations is homogenous in l , m and n . In order to obtain a nontrivial solution of the directions l , m and n from Equation (3.12d), the determinant of the co-efficients should be zero.

$$\text{i.e., } \begin{vmatrix} (\varepsilon_x - \varepsilon) & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & (\varepsilon_y - \varepsilon) & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & (\varepsilon_z - \varepsilon) \end{vmatrix} = 0$$

Expanding the determinant of the co-efficients, we get

$$\varepsilon^3 - J_1\varepsilon^2 + J_2\varepsilon - J_3 = 0 \tag{3.12e}$$

where

$$J_1 = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$J_2 = \begin{vmatrix} \varepsilon_x & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_y \end{vmatrix} + \begin{vmatrix} \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{zy} & \varepsilon_z \end{vmatrix} + \begin{vmatrix} \varepsilon_z & \varepsilon_{zx} \\ \varepsilon_{xz} & \varepsilon_x \end{vmatrix}$$

$$J_3 = \begin{vmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_z \end{vmatrix}$$

We can also write as

$$J_1 = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$J_2 = \varepsilon_x\varepsilon_y + \varepsilon_y\varepsilon_z + \varepsilon_z\varepsilon_x - \frac{1}{4}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2)$$

$$J_3 = \varepsilon_x\varepsilon_y\varepsilon_z + \frac{1}{4}(\gamma_{xy}\gamma_{yz}\gamma_{zx} - \varepsilon_x\gamma_{yz}^2 - \varepsilon_y\gamma_{zx}^2 - \varepsilon_z\gamma_{xy}^2)$$

Hence the three roots $\varepsilon_1, \varepsilon_2$ and ε_3 of the cubic Equation (3.12e) are known as the principal strains and J_1, J_2 and J_3 are termed as first invariant, second invariant and third invariant of strains, respectively.

Invariants of Strain Tensor

These are easily found out by utilizing the perfect correspondence of the components of strain tensor ε_{ij} with those of the stress tensor τ_{ij} . The three invariants of the strain are:

$$J_1 = \varepsilon_x + \varepsilon_y + \varepsilon_z \quad (3.13)$$

$$J_2 = \varepsilon_x \varepsilon_y + \varepsilon_y \varepsilon_z + \varepsilon_z \varepsilon_x - \frac{1}{4} (\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2) \quad (3.14)$$

$$J_3 = \varepsilon_x \varepsilon_y \varepsilon_z + \frac{1}{4} (\gamma_{xy} \gamma_{yz} \gamma_{zx} - \varepsilon_x \gamma_{yz}^2 - \varepsilon_y \gamma_{zx}^2 - \varepsilon_z \gamma_{xy}^2) \quad (3.15)$$

3.1.10 OCTAHEDRAL STRAINS

The strains acting on a plane which is equally inclined to the three co-ordinate axes are known as octahedral strains. The direction cosines of the normal to the octahedral plane are,

$$\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}.$$

The normal octahedral strain is:

$$\begin{aligned} (\varepsilon_n)_{oct} &= \varepsilon_1 l^2 + \varepsilon_2 m^2 + \varepsilon_3 n^2 \\ \therefore (\varepsilon_n)_{oct} &= \frac{1}{3} (\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \end{aligned} \quad (3.16)$$

$$\begin{aligned} \text{Resultant octahedral strain} &= (\varepsilon_R)_{oct} = \sqrt{(\varepsilon_1 l)^2 + (\varepsilon_2 m)^2 + (\varepsilon_3 n)^2} \\ &= \sqrt{\frac{1}{3} (\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2)} \end{aligned} \quad (3.17)$$

$$\text{Octahedral shear strain} = \gamma_{oct} = \frac{2}{3} \sqrt{(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2} \quad (3.18)$$